WMML
Meet #2
Nov. 12, 2019

Name _____

School _____

Arithmetic and Number Theory

1) Find the units digit of $2019^{11} + 2019^{12}$.

1._____0_____

Units digit alternates between 9 and 1. The 11^{th} power lands on 9 and the 12^{th} power lands on 1, so the sum ends in a zero.

2) Find the units digit of $2019(7^{2019} - 6^{2019})^{2019}$.

2._____7_____

Units digit for powers of 7 is cyclic with {7, 9, 3, 1}, landing on 3.

Units digit for powers of 6 always ends in a 6.

$$3 - 6 = 7\ (modulo\ 10)$$

Units digit for powers of 7 is cyclic with {7, 9, 3, 1}, landing on 3.

$$9(3) = 27$$

3) Let x be a six-digit number whose first three digits are 523 such that the integer is divisible by each of 7, 8, and 9. Find the sum of all possible values of x.

3._____1,046,808_____

$523ABC$ needs to be a multiple of $lcm(7,8,9) = 504$.

$523ABC - 504000 = 19ABC$ must also be a multiple of 504.

$\frac{19000}{500} = 38$ gives us a place to start looking.

$38(504) = 19152$ and $39(504) = 19656$

$(19152 + 504000) + (19656 + 504000) = 1046808$

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Algebra 1

1) The sum of the squares of the roots of the equation $x^2 + 2hx = 3$ is 10. Find $|h|$.

1._____1_____

$$x^2 + 2hx - 3 = 0$$

$$(x-3)(x+1) = 0 \quad \text{or} \quad (x+3)(x-1) = 0$$

$$2h = -2 \quad \text{or} \quad 2h = 2$$

$$|h| = 1$$

2) Simplify $\sqrt{4+\sqrt{7}} - \sqrt{4-\sqrt{7}}$.

2._____$\sqrt{2}$_____

$$x = \sqrt{4+\sqrt{7}} - \sqrt{4-\sqrt{7}}$$

$$x^2 = \left(4+\sqrt{7}\right) + \left(4-\sqrt{7}\right) - 2\left(\sqrt{4+\sqrt{7}}\right)\left(\sqrt{4-\sqrt{7}}\right)$$

$$= 8 - 2\left(\sqrt{4+\sqrt{7}}\right)\left(\sqrt{4-\sqrt{7}}\right) = 8 - 2\sqrt{(4+\sqrt{7})(4-\sqrt{7})}$$

$$= 8 - 2\sqrt{9} = 2$$

$$x = \sqrt{2}$$

3) AfterMath Airlines currently charges $200 per ticket, and sells 40,000 tickets in a year. For every $10 they increase the ticket price, they sell 1000 fewer tickets. How much should they charge to maximize their revenue?

3._____$300_____

Price: $200 + 10n = 10(20+n)$

Tickets: $40000 - 1000n = 1000(40-n)$

Revenue:

$$10(20+n)(1000)(40-n) = 10000(-n^2 + 20n + 800)$$

Max when $n = -\frac{20}{2(-1)} = 10$, so price = $200 + 10(10) = 300$

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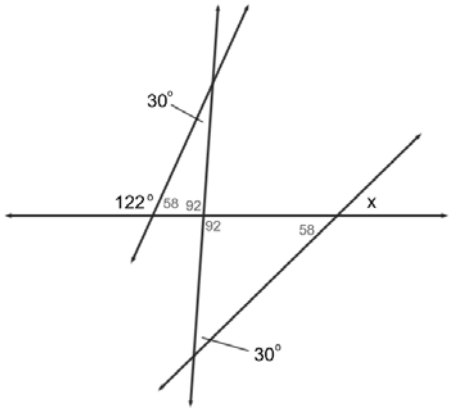
Geometry

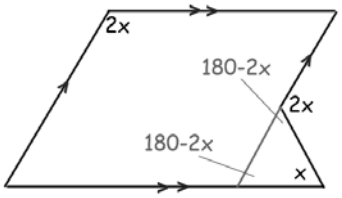
1) What is the value of x in the diagram below?

1._____58°_____



2) What is the value of x in the diagram below?

2._____60°_____



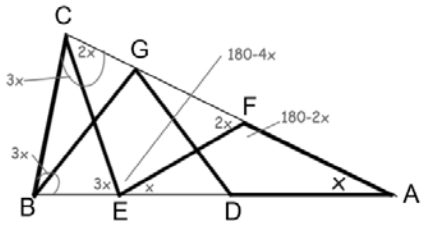
$$(180 - 2x) + (180 - 2x) + x = 180$$

$$x = 60$$

3) In ΔABC in the diagram below, D and E are points on side $\overline{AB}$, and F and G are points on side $\overline{AC}$, such that $AD = DG = GB = BC = CE = EF = FA$. Find the value of x.

$\frac{180}{7}$ *or* $25\frac{5}{7}$

3._____



$$3x + 3x + x = 180$$

$$x = \frac{180}{7}$$

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Algebra 2

1) Find the value of k if 3 is a root of $f(x) = x^3 + 3x^2 + kx - 9$.

1._____-15_____

$$f(3) = 3^3 + 3(3)^2 + k(3) - 9$$
$$0 = 27 + 27 - 9 + 3k$$
$$3k = -45$$
$$k = -15$$

2) What is the sum of all possible values of c for which the equation $\frac{x^2-x+c}{x^2-8x+15}$ has exactly one vertical asymptote?

2._____-26_____

$$x^2 - 8x + 15 = (x-3)(x-5)$$

3 or 5 needs to be a root of the numerator

$$3^2 - 3 + c = 0 \qquad 5^2 - 5 + c = 0$$
$$6 + c = 0 \qquad 20 + c = 0$$
$$c = -6 \qquad c = -20$$

3) Find the sum of the real roots of the equation

3._____-4_____

$$(2 + (2 + (2 + (2 + x)^2)^2)^2)^2 = 15129.$$

$$2 + (2 + (2 + (2 + x)^2)^2)^2 = 123$$
$$2 + (2 + (2 + x)^2)^2 = 11$$
$$2 + (2 + x)^2 = 3$$
$$2 + x = \pm 1$$
$$x = -1 \text{ or } x = -3$$
$$-1 + (-3) = -4$$

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Trigonometry and Complex Numbers

1) Let ΔABC be a right triangle with $\angle ABC = 90°$, $AB = 12$, and $AC = 20$. Find the value of

$$\frac{\sin(A)\,cos(A)}{\tan(A)}.$$

$\frac{9}{25}$

1._____

$\sin(A) = \frac{16}{20} = \frac{4}{5}$ $\cos(A) = \frac{12}{20} = \frac{3}{5}$ $\tan(A) = \frac{16}{12} = \frac{4}{3}$

$$\left(\frac{4}{5}\right)\left(\frac{3}{5}\right)\left(\frac{3}{4}\right) = \frac{9}{25}$$

2) Four complex numbers lie at the vertices of a square in the complex plane. Three of the numbers are $1 + 2i$, $-2 + i$, and $-1 - 2i$. What is the fourth number?

$2 - i$

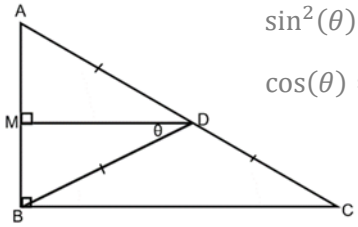
2._____

The midpoint of $1 + 2i$ and $-1 - 2i$ is $(0,0)$. Rotating $-2 + i$ by 180 degrees around $(0,0)$ gives the remaining vertex as $2 - i$.

3) Right triangle ABC has $\angle ABC = 90°$. Let the midpoint of $\overline{AB}$ be M. The perpendicular bisector of $\overline{AB}$ intersects $\overline{AC}$ at D, and $\sin(\angle MDB) = \frac{1}{5}$. If $MD = 1$, then what is AC?

$\frac{5\sqrt{6}}{6}$

3._____



$$\sin^2(\theta) + \cos^2(\theta) = 1$$

$$\cos(\theta) = \sqrt{1 - \left(\frac{1}{5}\right)^2} = \sqrt{\frac{24}{25}} = \frac{2\sqrt{6}}{5}$$

$$\cos(\theta) = \frac{1}{BD}$$

$$BD = \frac{1}{\frac{2\sqrt{6}}{5}} = \frac{5}{2\sqrt{6}} = \frac{5\sqrt{6}}{12}$$

$$AC = 2BD = \frac{5\sqrt{6}}{6}$$

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Precalculus

1) Find all values of a for which the line $y = x + a$ passes through the vertex of the parabola $y = x^2 + a^2$.

1.____a=0 or a=1_____

Vertex lies on the y-axis. We need $x^2 + a^2 = x + a$ for $x = 0$.

$$0^2 + a^2 = 0 + a$$
$$a^2 = a$$
$$a = 0 \quad \text{or} \quad a = 1$$

2) Find the area of the region enclosed by the graph of the equation $y = \frac{(x-4)^2}{12} + \frac{(y+2)^2}{12} = 1$.

12π

2._____

The equation represents a circle centered at $(4, -2)$ with a radius of $\sqrt{12}$.

$$Area = \pi\left(\sqrt{12}\right)^2 = 12\pi$$

3) A set of n consecutive positive integers beginning with 1 is written on a blackboard. One number is erased, and the average of the remaining $n - 1$ integers is $35\frac{7}{17}$. What number is erased?

3._____7_____

n needs to be around 70, and $n - 1$ needs to be a multiple of 17, so $n = 69$.

$$68\left(35\frac{7}{17}\right) = 2408$$
$$\frac{69(70)}{2} = 2415$$
$$2415 - 2408 = 7$$

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Team Round

1) Find the value of n if 2^n is the highest power of 2 that evenly divides $30!$.

Number of factors that are multiples of 2: 15

Number of factors that are multiples of 4: 7

Number of factors that are multiples of 8: 3

Number of factors that are multiples of 16: 1

$$15 + 7 + 3 + 1 = 26$$

1. 26

2) Find the real values of K for which the equation $x = K^2(x-1)(x-2)$ has real roots.

$$K^2x^2 + (-3K^2 - 1)x + 2K^2 = 0$$

$$b^2 - 4ac = (-3K^2 - 1)^2 - 4(K^2)(2K^2)$$
$$= 9K^4 + 6K^2 + 1 - 8K^4$$

$$K^4 + 6K^2 + 1 \geq 0$$

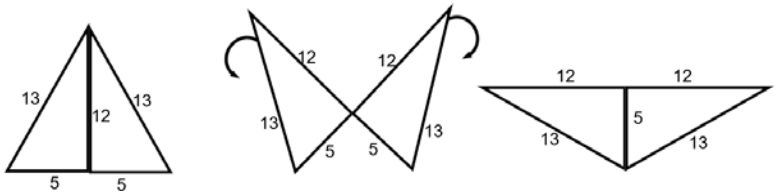
Since K is real, this is true for all values of K.

2. All real numbers

3. 24

4. x=5 or x=23

3) A triangle has sides measuring 13 cm, 13 cm, and 10 cm. A second triangle is drawn with sides measuring 13 cm, 13 cm, and x cm, where x is a whole number other than 10. If the two triangles have equal areas, what is the value of x?



5. $\frac{\pi}{6}$

6. 2

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4) Find all values of x such that $2\log_3(x+4) - \log_3(4x-11) = 2$.

$$\log_3\left(\frac{(x+4)^2}{4x-11}\right) = 2$$

$$\frac{(x+4)^2}{4x-11} = 9$$

$$x^2 - 28x + 115 = (x-5)(x-23) = 0$$

$$x = 5 \quad \text{or} \quad x = 23$$

5) Find all acute angles θ such that

$$\sin(\theta) + \sin(2\theta) = \cos(\theta) + \cos(2\theta).$$

$$\sin^2(\theta) - 2\sin(\theta)\cos(\theta) + \cos^2(\theta) = \cos^2(2\theta) - 2\cos(2\theta)\sin(2\theta) + \sin^2(2\theta)$$

$$1 - 2\sin(\theta)\cos(\theta) = 1 - 2\cos(2\theta)\sin(2\theta)$$

$$\sin(\theta)\cos(\theta) = \cos(2\theta)\sin(2\theta)$$

$$\sin(\theta)\cos(\theta) = 2\cos(2\theta)\sin(\theta)\cos(\theta)$$

$$2\cos(2\theta)\sin(\theta)\cos(\theta) - \sin(\theta)\cos(\theta) = 0$$

$$\sin(\theta)\cos(\theta)\,(2\cos(2\theta) - 1) = 0$$

$$2\cos(2\theta) - 1 = 0$$

$$\theta = \frac{\pi}{6}$$

6) Find r if r is positive and the line whose equation is $x + y = r$ is tangent to the circle whose equation is $x^2 + y^2 = r$.

The tangent point has to be at $\left(\frac{r}{2}, \frac{r}{2}\right)$. Since this is a point on the circle we can plug it into the equation of the circle:

$$\left(\frac{r}{2}\right)^2 + \left(\frac{r}{2}\right)^2 = r$$

$$\frac{r^2}{2} = r$$

$$r^2 - 2r = 0$$

$$r(r-2) = 0$$

$$r = 2$$