WMML
Meet #5
Feb. 5, 2019

Name _____

School _____

Arithmetic and Number Theory

1) How many positive integer divisors of 30! are prime?

1._____10_____

There are ten prime numbers less than or equal to 30:

2, 3, 5, 7, 11, 13, 17, 19, 23, 29

2) A palindrome is an integer that reads the same forward as it does backward, such as 12321. Find the total number of ten-digit palindromes.

2._____90,000_____

(9)(10)(10)(10)(10)(1)(1)(1)(1)(1) = 90,000

The first digit cannot be zero and once the first 5 digits are chosen the last 5 are already chosen as well.

3) How many positive integer divisors of 5400 are **not** multiples of any perfect square greater than 1?

3._____8_____

$$5400 = 2^3 \cdot 3^3 \cdot 5^2$$

We cannot have any square factors, so each prime factor can only have an exponent of 0 or 1.

$$2 \cdot 2 \cdot 2 = 8$$

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Algebra 1

$\left(-1, \frac{5}{2}\right)$

1._____

1) Find all ordered pairs (a, b) such that $3a - 2b = -8$ and $5a + 4b = 5$.

$$\begin{array}{r} 5a + 4b = 5 \\ 6a - 4b = -16 \\ \hline 11a \quad = -11 \end{array}$$

$a = -1$

$5(-1) + 4b = 5$

$4b = 10$

$b = 5/2$

2) Let x and y be real numbers satisfying $\frac{2}{x} = \frac{y}{3} = \frac{x}{y}$.
Determine x^3.

2._____12_____

$$\frac{2}{x} = \frac{x}{y} \rightarrow y = \frac{x^2}{2}$$
$$\frac{y}{3} = \frac{x}{y} \rightarrow y^2 = 3x$$
$$\left(\frac{x^2}{2}\right)^2 = 3x$$
$$x^3 = 12$$

(3,0) and $\left(-\frac{9}{5}, \frac{12}{5}\right)$

3) Find all pairs of real numbers (a, b) such that $(x - a)^2 + (2x - b)^2 = (x - 3)^2 + (2x)^2$ for all x.

3._____

$$x^2 - 2ax + a^2 + 4x^2 - 4bx + b^2 = x^2 - 6x + 9 + 4x^2$$
$$(-2a - 4b)x + (a^2 + b^2) = -6x + 9$$
$$\begin{cases} -2a - 4b = -6 \\ a^2 + b^2 = 9 \end{cases}$$

(3,0) and $\left(-\frac{9}{5}, \frac{12}{5}\right)$

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Geometry

1) $ABCDEFGH$ is a right rectangular prism with volume 672 in^3. If $AB = 6\ in$ and $BC = 8\ in$, then what is length of the third dimension?

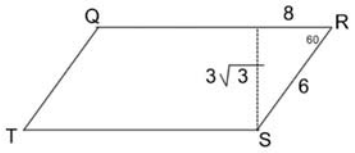
1._____14_____

$$(6)(8)x = 672$$
$$x = 14$$

2) Base $IJKL$ of right prism $IJKLMNOP$ is a parallelogram with $IJ = 8$, $JK = 6$, and $\angle IJK = 60°$. Given that the height of the prism is 9, find the surface area of $IJKLMNOP$.

$252 + 48\sqrt{3}$

2._____



$$2(8)\left(3\sqrt{3}\right) + 2(6)(9) + 2(8)(9) = 252 + 48\sqrt{3}$$

3) $QRSTUVWX$ is a cube. How many different planes pass through at least three vertices of $QRSTUVWX$?

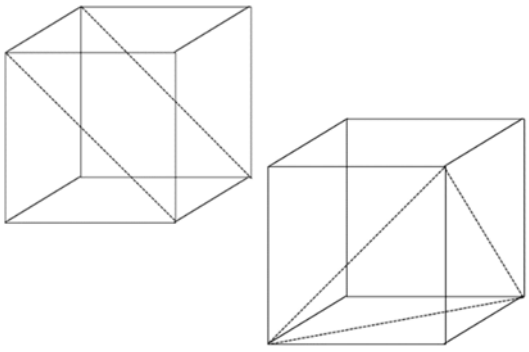
3._____20_____

Faces (4 vertices): 6

Edges to opposite edge (4 vertices): 6

Each vertex can be "chopped off" (3 vertices): 8

$$6 + 6 + 8 = 20$$



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Algebra 2

1) Solve the inequality $x^2 - 12x + 32 \leq 0$.

$4 \leq x \leq 8$

1._____

$$(x-4)(x-8) \leq 0$$

$(-\infty, 4)$: positive

$(4,8)$: negative

$(8, \infty)$: positive

2) The polynomial $f(x) = x^4 + ax^3 + bx^2 + cx + d$ has roots 1, 3, 5, and 7. Determine $a + b + c + d$.

2._____-1_____

$$(x-1)(x-3)(x-5)(x-7) = x^4 - 16x^3 + 86x^2 - 176x + 105$$

$$-16 + 86 - 176 + 105 = -1$$

3) There is a unique polynomial $P(x)$ of the form

$$P(x) = 7x^7 + bx^6 + cx^5 + \cdots + gx + h$$

such that $P(1) = 1, P(2) = 2, \ldots, P(7) = 7$. Find $P(0)$.

3._____-35,280_____

$P(x) - x = 0$ for $x = 1, 2, 3, 4, 5, 6, 7$

$P(x) - x = a(x-1)(x-2)(x-3)\cdots(x-7)$

Since $a = 7$, we have $P(x) = 7(x-1)(x-2)\cdots(x-7) + x$

So $P(0) = 7(0-1)(0-2)(0-3)\cdots(0-7) + 0 = -35{,}280$

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Trigonometry and Complex Numbers

$-\sqrt{2}$

1) Evaluate $\cos(225°) + \sin(225°)$.

1._____

$$\cos(225°) = \sin(225°) = -\frac{\sqrt{2}}{2}$$
$$-\frac{\sqrt{2}}{2} + \left(-\frac{\sqrt{2}}{2}\right) = -\sqrt{2}$$

2) The complex number z satisfies $z + |z| = 2 + 8i$. What is $|z|$?

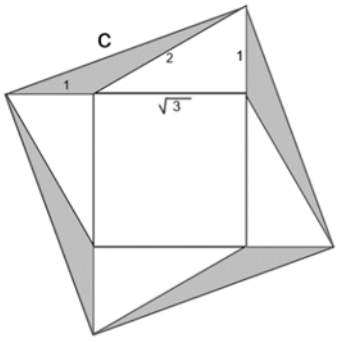
2._____17_____

$$a + bi + \sqrt{a^2 + b^2} = 2 + 8i$$
$$b = 8$$
$$a + \sqrt{a^2 + b^2} = 2$$
$$a = -15$$
$$|-15 + 8i| = \sqrt{225 + 64} = 17$$

$\frac{10 - 4\sqrt{3}}{13}$

3) Four congruent 30-60-90 triangles are constructed on the sides of a square as shown below. The hypotenuse of each of these triangles has length 2. The outer vertices of these triangles are connected to form quadrilateral $ABCD$. What fraction of $ABCD$ is shaded?

3._____



Large square:

$$c^2 = 1^2 + 2^2 - 2(1)(2)\cos(150°)$$
$$c^2 = 5 + 2\sqrt{3}$$

White region:

$$3 + 4\left(\frac{1}{2}\right)(\sqrt{3})(1) = 3 + 2\sqrt{3}$$

Fraction:

$$\frac{(5 + 2\sqrt{3}) - (3 + 2\sqrt{3})}{5 + 2\sqrt{3}} = \frac{2}{5 + 2\sqrt{3}} = \frac{10 - 4\sqrt{3}}{13}$$

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Precalculus

1) Determine the value of

$$\sum_{k=1}^{8}(2k-1)$$

1._____64_____

$$1+3+5+7+9+11+13+15$$
$$=16(4)=64$$

2) A line $x=k$ intersects the graph of $y=\log_5(x)$ and the graph of $y=\log_5(x+4)$. The distance between the points of intersection is 0.5. Find k.

2._____$1+\sqrt{5}$_____

$$\log_5(x+4)-\log_5(x)=0.5$$
$$\log_5\left(\frac{x+4}{x}\right)=\frac{1}{2}$$
$$\sqrt{5}=\frac{x+4}{x}$$
$$x=\frac{4}{-1+\sqrt{5}}=1+\sqrt{5}$$

3) An ellipse has foci at $(0,0)$ and $(14,0)$ and passes through the vertex of the parabola with equation $y=x^2-10x+37$. Find the length of the major axis of the ellipse.

3._____28_____

The vertex of the parabola is at $x=\frac{10}{2}$, so the vertex is $(5,12)$.

The length of the major axis is equal to the sum of the distances from each foci to any point on the ellipse:

$$\sqrt{(5-0)^2+(12-0)^2}+\sqrt{(14-5)^2+(0-12)^2}$$
$$=\sqrt{169}+\sqrt{225}=13+15=28$$