WMML
Meet #4
Jan. 8, 2019

Name ____________________

School ____________________

Arithmetic and Number Theory

1) What is the largest integer whose cube is less than 10,000?

1. ___21___

$$21^3 = 9261$$
$$22^3 = 10648$$

2) What is the greatest common factor of the 10 smallest positive common multiples of the numbers 30 and 50?

2. ___150___

This will be the least common multiple of 30 and 50, which is 150.

3) A set of tiles numbered from 1 through 100 is modified repeatedly by the following operation: remove all tiles that are perfect cubes, and re-number all remaining tiles starting with 1. How many times must this operation be performed to reduce the number of tiles to 1?

3. ___36___

Cubes less than 100: 1, 8, 27, 64

The number of tiles will reduce by 4 until there are less than 64, so 10 times. Then it will reduce by 3 until less than 27, so 12 times. Reduce by 2 a total of 9 times, then by 1 for the remaining 5 times.

$$10 + 12 + 9 + 5 = 36$$

WMML
Meet #4
Jan. 8, 2019

Name ______________________

School _____________________

Algebra 1

1) The population of a Masschusetts town increased by 25% during 2018. By what percent will it need to decrease during 2019 to return to the population it was at the beginning of 2018?

1. 20%

$$p_{19} = \left(\frac{5}{4}\right)p_{18}$$
$$p_{19}\left(\frac{4}{5}\right) = p_{18}$$

Decrease by 20%.

2) Pipe A can fill a pool in 5 hours, while pipe B can fill it in four. How many hours will it take to fill the pool if both are operating at the same time?

$$\frac{20}{9} = 2\frac{2}{9}$$

2. ______________________

Working together, the pipes will fill the pool at a rate of $\frac{1}{5} + \frac{1}{4} = \frac{9}{20}$ of the pool per hour. The pool will be full in $\frac{20}{9}$ hours.

3) Adam has $3.08 in pennies, nickels, and quarters. He has four more pennies than quarters and one more nickel than pennies. How many coins does he have?

3. 36

$$.01p + .05n + .25q = 3.08$$
$$q = p - 4$$
$$n = p + 1$$
$$.01p + .05(p+1) + .25(p-4) = 3.08$$
$$.01p + .05p + .25p = 4.03$$
$$.31p = 4.03$$
$$p = 13, q = 9, n = 14$$

WMML
Meet #4
Jan. 8, 2019

Name ______________________

School _____________________

Geometry

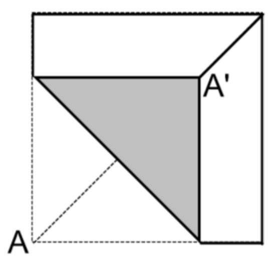
1) The area of a trapezoid is 96. One base is 6 units longer than the other, and the height of the trapezoid is 8. Find the length of the shorter base.

1. 9

$$A = \frac{1}{2}(b + b + 6)(8)$$
$$2b + 6 = 24$$
$$b = 9$$

2) A square sheet of paper has an area of 6 cm^2. The front is white and the back is shaded. When the sheet is folded so that point A rests on the diagonal as shown, the visible shaded area is equal to the visible white area. How many centimeters is A' from its original position, A?

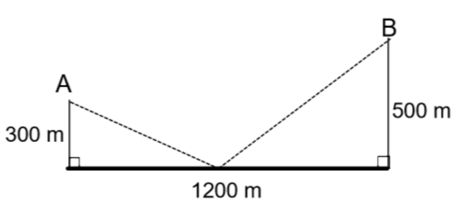
2. $2\sqrt{2}$



Shaded area is 6/3 = 2, so the legs of the shaded triangle are 2 cm. Therefore $\overline{AA'} = \sqrt{2^2 + 2^2} = \sqrt{8} = 2\sqrt{2}$

3) The rules of a race require that all runners start at A, touch any part of the 1200-meter wall, and stop at B. What is the number of meters in the minimum distance a participant must run?

3. $400\sqrt{13}$



If we reflect point B over the wall to get B’, then the minimum distance will be the length of the segment from A to B’.

$$D = \sqrt{(800)^2 + (1200)^2} = \sqrt{640000 + 1440000} = 400\sqrt{13}$$

WMML
Meet #4
Jan. 8, 2019

Name ______________________

School _____________________

Algebra 2

1) Find all x such that $-4 < \frac{1}{x} < 3$.

$x > \frac{1}{3}$ or $x < -\frac{1}{4}$

1.______________________

If x is positive: $-4x < 1 < 3x$ *so* $x > -\frac{1}{4}$ ***and*** $x > \frac{1}{3}$

Overlap is $x > \frac{1}{3}$

If x is negative: $-4x > 1 > 3x$ *so* $x < -\frac{1}{4}$ ***and*** $x < \frac{1}{3}$

Overlap is $x < -\frac{1}{4}$

$$\frac{2x}{x+2}$$

2.______________________

2) Let $f(x) = \frac{4x}{x+2}$ and $g(x) = \frac{2x}{x+4}$. Find $f\big(g(x)\big)$ in simplest terms.

$$f\big(g(x)\big) = \frac{4(\frac{2x}{x+4})}{\left(\frac{2x}{x+4}\right) + 2} = \frac{\frac{8x}{x+4}}{\frac{4x+8}{x+4}} = \frac{8x}{4x+8} = \frac{2x}{x+2}$$

3.__________5__________

3) Solve for x:

$$\sqrt{x + \sqrt{x+11}} + \sqrt{x - \sqrt{x+11}} = 4.$$

$$\left(x + \sqrt{x+11}\right) + 2\sqrt{x + \sqrt{x+11}}\sqrt{x - \sqrt{x+11}} + \left(x - \sqrt{x+11}\right) = 16$$

$$2x + 2\sqrt{x^2 - (x+11)} = 16$$

$$\sqrt{x^2 - x - 11} = (8 - x)$$

$$x^2 - x - 11 = 64 - 16x + x^2$$

$$15x = 75$$

$$x = 5$$

WMML
Meet #4
Jan. 8, 2019

Name ______________________

School _____________________

Trigonometry and Complex Numbers

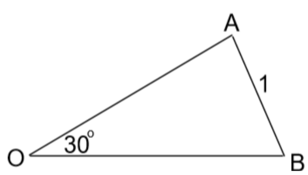
1) Let $a = 3 + 4i$ and $b = 12 - 5i$. In $a + bi$ form, what is the value of $a^2 + 3b + 2$?

1.______ $31 + 9i$ ______

$$(3 + 4i)^2 + 3(12 - 5i) + 2$$
$$= 9 + 24i - 16 + 36 - 15i + 2$$
$$= 31 + 9i$$

2) Two rays with common endpoint O form a $30°$ angle. Point A lies on one ray, point B on the other ray, and $\overline{AB} = 1$. What is the maximum possible length of $\overline{OB}$?

2.__________2__________



$$\frac{OB}{\sin(A)} = \frac{AB}{\sin(O)} = \frac{1}{\sin(30°)} = 2$$
$$OB = 2\sin(A)$$

To maximize OB we need $\sin(A) = 1$, so $A = 90°$. So ΔOBA is a 30-60-90 triangle and $OB = 2$.

3.__________4__________

3) How many triangles have area 10 and vertices $(-5,0)$, $(5,0)$, and $(5\cos(\theta), 5\sin(\theta))$ for some angle θ?

$$10 = \frac{1}{2}(10)(5\sin(\theta))$$
$$\sin(\theta) = \pm\frac{2}{5}$$

There are 4 unique angles θ on the unit circle that satisfy this equation, so there are 4 triangles.

WMML
Meet #4
Jan. 8, 2019

Name ______________________

School _____________________

Precalculus

1) Let $f(x) = 2x - 4$. What is $f^{-1}(f^{-1}\left(f\left(f\left(f^{-1}(2)\right)\right)\right))$?

1.__________3__________

$$f^{-1}\left(f^{-1}\left(f\left(f\left(f^{-1}(2)\right)\right)\right)\right) = f^{-1}(2)$$
$$2 = 2x - 4$$
$$x = 3$$

2) Suppose the ordered pair (x, y) satisfies $\frac{\log_{10}(xy)}{\log_{10}(\frac{x}{y})} = \frac{1}{2}$. If y is increased by 50%, by what fraction must x be multiplied to keep this equation true?

2.__________ $\frac{8}{27}$ __________

$$2\log_{10}(xy) = \log_{10}\left(\frac{x}{y}\right)$$
$$x^2y^2 = \frac{x}{y}$$
$$xy^3 = 1$$

If y is multiplied by $\frac{3}{2}$, then x must be multiplied by $\left(\frac{2}{3}\right)^3 = \frac{8}{27}$ to keep it equal.

3) Find the largest value of x for which $x^2 + y^2 = x + y$ has a solution, if x and y are real.

3.__________ $\frac{1+\sqrt{2}}{2}$ __________

$$\left(x^2 - x + \frac{1}{4}\right) + \left(y^2 - y + \frac{1}{4}\right) = \frac{1}{2}$$
$$\left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{2}$$

This is a circle centered at $(\frac{1}{2}, \frac{1}{2})$ with radius $\frac{\sqrt{2}}{2}$. The largest possible value of x is at $\frac{1}{2} + \frac{\sqrt{2}}{2} = \frac{1+\sqrt{2}}{2}$

WMML
Meet #4
Jan. 8, 2019

Name ______________________

School _____________________

Team Round

1) The LCM of a pair of positive integers is 504, and the GCF of the numbers is 6. One of the numbers is 12. What is the other number?

$$No\ solution$$

2) A town's population increased by 1200 people, and then this new population decreased by 11%. The town now had 32 less people than it did before the 1200 increase. What was the original population?

$$(p + 1200)(.89) = p - 32$$
$$.89p + 1068 = p - 32$$
$$.11p = 1100$$
$$p = 10000$$

3) A circle is inscribed in a large square and circumscribed about a smaller square. The area of the larger square is 8 square meters. What is the area of the smaller square?

Side of large square = Diameter of circle = Diagonal of small square $= \sqrt{8} = 2\sqrt{2}$.

$$a^2 + a^2 = \left(2\sqrt{2}\right)^2$$
$$2a^2 = 8$$
$$a = 2$$

1)______No solution_______

2)________10000_________

3)__________2___________

4)________128000________

5) $\frac{3}{10}$

6) $-\frac{22}{25}$

WMML
Meet #4
Jan. 8, 2019

Name ______________________

School _____________________

4) Find the value of $a^3b^7c^{14}$ given that $a^3b^2c = 108$ and $a^2b^3c^5 = 240$.

$$a^3b^7c^{14} = (a^3b^2c)^x(a^2b^3c^5)^y = a^{3x+2y}b^{2x+3y}c^{x+5y}$$

$$\begin{cases} 3x + 2y = 3, \\ 2x + 3y = 7, \\ x + 5y = 14. \end{cases}$$

Solving this system gives $(x, y) = (-1,3)$, so $a^3b^7c^{14} = (a^3b^2c)^{-1}(a^2b^3c^5)^3 = \frac{240^3}{108} = 128000$

5) If $\sin(x) = 3\cos(x)$, then what is $(\sin(x))(\cos(x))$?

$$\sin^2(x) + \cos^2(x) = 1$$
$$9\cos^2(x) + \cos^2(x) = 1$$
$$\cos(x) = \pm\frac{1}{\sqrt{10}}$$
$$\sin(x) = 3\cos(x) = \pm\frac{3}{\sqrt{10}}$$

Either way, $\sin(x)\cos(x) = \frac{3}{10}$.

6) Find all t such that $2\log_3(1 - 5t) = \log_3(2t + 5) + 2$.

$$2\log_3(1 - 5t) - \log_3(2t + 5) = 2$$
$$\log_3\left(\frac{(1 - 5t)^2}{2t + 5}\right) = 2$$
$$3^2 = \frac{(1 - 5t)^2}{2t + 5}$$
$$18t + 45 = 1 - 10t + 25t^2$$
$$25t^2 - 28t - 44 = 0$$
$$t = -\frac{22}{25}, t = 2$$

($t = 2$ is extraneous)