Arithmetic and Number Theory

1)  When $n$ is divided by 5, the remainder is 1. What is the remainder when $3n$ is divided by 5?

1._____3_____

$$n = 5k + 1$$
$$3n = 3(5k + 1) = 5(3k) + 3$$

So the remainder is 3.

2)  What is the units digit of $7^{42} + 42^7$?

2._____7_____

The units digit for powers of 7 follow the pattern:
7,9,3,1,7,9,3,1,…

The units digit for powers of 42 follow the pattern:
2,4,8,6,2,4,8,6,…

The units digit of $7^{42} + 42^7$ is determined by $9 + 8 = 17$, so the units digit is 7.

3)  The number 3456 is currently a base 10 number, what is this number in base 7?

3._____13035_____

$$3456 = 1(2401) + 3(343) + 0(49) + 3(7) + 5(1)$$

Algebra 1

1) If $j^k = 5$, then what is the value of $j^{2k} + j^{3k} + j^{4k}$ ?

1._____775_____

$$j^{2k} + j^{3k} + j^{4k} = \left(j^k\right)^2 + \left(j^k\right)^3 + \left(j^k\right)^4$$
$$= 5^2 + 5^3 + 5^4 = 25 + 125 + 625 = 775$$

2) The solution to the equation $40x^3 = 625$ can be written in the form $\frac{a}{b}$ where $a$ and $b$ are relatively prime. What is the value of $(ab)^3$?

2._____1000_____

$$x^3 = \frac{625}{40} = \frac{125}{8}$$
$$x = \frac{5}{2}$$
$$(ab)^3 = (5 \cdot 2)^3 = 10^3 = 1000$$

3) A boat travelled 3360 miles across the ocean from Boston to Spain. If the boat had travelled 10 mph faster, it would have made it to Spain 56 hours sooner.  How fast was the boat travelling in miles per hour?

3._____20 mph_____

$$3360 = rt \quad \rightarrow \quad t = \frac{3360}{r}$$
$$3360 = (r + 10)(t - 56)$$
$$3360 = (r + 10)\left(\frac{3360}{r} - 56\right)$$
$$0 = -56r^2 - 560r + 33600$$
$$= r^2 + 10r - 600 = (r + 30)(r - 20)$$
$$r = 20 \text{ mph}$$

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Geometry

1) How many scalene triangles have all sides that are integer length and a perimeter less than 10?

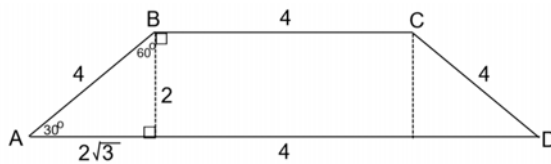There are seven integer triples with sums less than 10, but 2,3,4 forms the only valid triangle.

1._____1_____

2) In parallelogram $ABCD$ it is known that $\angle A = 3x + 20°$ and $\angle C = 40 - x°$.  What is the measure of $\angle B + \angle D$?

$$3x + 20 = 40 - x$$
$$x = 5$$
$$\angle A = \angle C = 3(5) + 20 = 35°$$
$$\angle B = \angle D = 145°$$
$$\angle B + \angle D = 290°$$

2._____290°_____

3) Given that $ABCD \cdots L$ is a regular dodecagon, find the length $AD$ if $AB = 4$.



$$\angle B = \frac{(12 - 2)(180)}{12} = 150°$$
$$\overline{AM} = 2\sqrt{3}$$
$$\overline{AD} = 4 + 4\sqrt{3}$$

3._____$4 + 4\sqrt{3}$_____

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Algebra 2

1) Find all $x$ such that $x^2 + x - 30 \geq 0$ and $\frac{6}{x} > 0$.

1._____$x > 5$_____

$(x + 6)(x - 5) \geq 0$   so   $x \leq -6$ or $x \geq 5$

Since $x > 0$ the only valid solutions are $x \geq 5$

2) Find a constant $c$ such that there is no remainder when $x^3 + cx^2 + 4x - 21$ is divided by $x - 3$.

2._____$c = -2$_____

$$
\begin{array}{r|rrrr}
3 & 1 & c & 4 & -21 \\
  &   & 3 & 3c+9 & 9c+39 \\
\hline
  & 1 & c+3 & 3c+13 & 9c+18
\end{array}
$$

$$9c + 18 = 0$$
$$c = -2$$

3) What is $\dfrac{\sqrt{2}}{5-\sqrt{2}-\sqrt{3}}$ when its denominator is rationalized?

$\dfrac{13+25\sqrt{2}+5\sqrt{3}+6\sqrt{6}}{94}$

3._____

$$\frac{\sqrt{2}}{5 - \sqrt{2} - \sqrt{3}} \cdot \frac{5 - \sqrt{2} + \sqrt{3}}{5 - \sqrt{2} + \sqrt{3}} = \frac{-2 + 5\sqrt{2} + \sqrt{6}}{24 - 10\sqrt{2}}$$

$$\frac{-2 + 5\sqrt{2} + \sqrt{6}}{24 - 10\sqrt{2}} \cdot \frac{24 + 10\sqrt{2}}{24 + 10\sqrt{2}} = \frac{13 + 25\sqrt{2} + 5\sqrt{3} + 6\sqrt{6}}{94}$$

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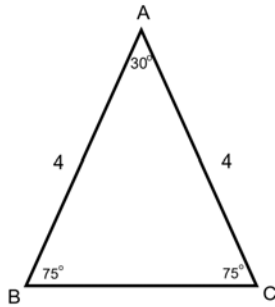Trigonometry and Complex Numbers

1) What is the value of $\tan(2910°)$?

$$2910 = 16(180) + 30$$

$$\tan(2910) = \tan(30) = \frac{\sin(30)}{\cos(30)} = \frac{\sqrt{3}}{3}$$

1.$\dfrac{\sqrt{3}}{3}$ _____

2) In isosceles triangle $ABC$, we are given that $AB = AC = 4$ and $\angle C = 75°$. Find the area of $\triangle ABC$.

2._____ 4 _____



$Area = \frac{1}{2}(\overline{AB})(\overline{AC})\sin(A)$

$$= \frac{1}{2}(4)(4)\left(\frac{1}{2}\right) = 4$$

3._____ $-1 - i$ _____

3) Find all complex numbers $z$ such that

$$|z - 1| = |z + 3| = |z - i|.$$

$z = a + bi$

$\sqrt{(a - 1)^2 + (b)^2} = \sqrt{(a + 3)^2 + (b)^2}$

$a^2 - 2a + 1 + b^2 = a^2 + 6a + 9 + b^2$

$a = -1$

$\sqrt{(-1 - 1)^2 + (b)^2} = \sqrt{(-1)^2 + ((b - 1))^2}$

$4 + b^2 = 1 + b^2 - 2b + 1$

$b = -1$

Precalculus

1) Given $a_0 = 1, a_1 = 3$, and the general relation
$a_n^2 - a_{n-1}a_{n+1} = (-1)^n$ for $n \geq 1$, find $a_3$.

1._____ 33 _____

$$3^2 - (1)(a_2) = (-1)^1$$
$$a_2 = 10$$
$$10^2 - (3)(a_3) = (-1)^2$$
$$a_3 = 33$$

2) Evaluate the expression $\log_6 2 + \log_6 27 + \log_6 4$.

2._____ 3 _____

$$= \log_6(2 \cdot 27 \cdot 4) = \log_6(216) = 3$$

$$\frac{3}{4}$$

3) If the line $y = mx + 1$ intersects the ellipse $x^2 + 4y^2 = 1$
exactly once, then find $m^2$.

3._____

$$x^2 + 4(mx + 1)^2 = 1$$
$$x^2 + 4m^2x^2 + 8mx + 4 = 1$$
$$(4m^2 + 1)x^2 + (8m)x + 3 = 0$$

Since there is only one solution we set discriminant equal to zero.

$$(8m)^2 - 4(4m^2 + 1)(3) = 0$$
$$m^2 = \frac{3}{4}$$

Team Round

1) A store sold 72 decks of cards for $\$a67.9b$. If every deck cost the same price, then what is the value of $a + b$?
($a$ and $b$ are both digits between 0 and 9)

1)_____5_____

Since the number is divisible by 8, then 79b must be divisible by 8. So $b = 2$.

Since the number is divisible by 9, then
$a + b + 6 + 7 + 9 = a + b + 22$ must be divisible by 9 so
$$a + b = 5$$

2) When there is no wind, a plane travelling at a constant rate can cover a round-trip from $A$ to $B$ and back in 10 hours. If the wind blows from $A$ towards $B$ at $k$ km/hr, it adds 25 minutes to the round trip. If $A$ and $B$ are 1000 km apart, then find $k$.

2)_____40_____

$$\frac{2000}{10} = 200\frac{km}{h}$$
$$10\frac{5}{12} = \frac{1000}{200 + k} + \frac{1000}{200 - k}$$
$$10\frac{5}{12}(12)(200 + k)(200 - k)$$
$$= 1000(12)(200 - k) + 1000(12)(200 + k)$$
$$(200 + k)(200 - k) = 96(200 - k) + 96(200 + k)$$
$$40000 - k^2 = 19200 - 96k + 19200 + 96k$$
$$k^2 = 1600$$
$$k = 40$$

3) A plane intersects a sphere, forming a circle. Find the radius of the circle if the radius of the sphere is 8 and the center of the sphere is 5 units from the plane.

3)_____$\sqrt{39}$_____
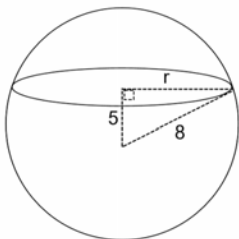


$$5^2 + r^2 = 8^2$$
$$r = \sqrt{39}$$

Team Round (Cont'd)

4) If $\begin{vmatrix} W & X \\ Y & Z \end{vmatrix} = 4$, then find $\begin{vmatrix} 4W & 4X \\ 4Y & 4Z \end{vmatrix}$.

4)_____64_____

$WZ - XY = 4$

$(4W)(4Z) - (4X)(4Y) = 16WZ - 16XY = 16(WZ - XY) = 64$

5) If $\sin(x) = \cos(2x)$ and $0 \le x \le \pi/2$, then find $x$.

5)_____$\pi/6$_____

$$\sin(x) = 1 - 2\sin^2(x)$$
$$2\sin^2(x) + \sin(x) - 1 = 0$$
$$(2\sin(x) - 1)(\sin(x) + 1) = 0$$
$$x = \frac{\pi}{6}$$

6)_____12_____

6) A parabola $y = ax^2 + bx + c$ has its vertex at (4,2). If (2,0) is on the parabola, then find $abc$.

$$y - 2 = a(x - 4)^2$$
$$0 - 2 = a(2 - 4)^2$$
$$a = -\frac{1}{2}$$
$$y = -\frac{1}{2}(x - 4)^2 + 2$$
$$y = -\frac{1}{2}x^2 + 4x - 6$$
$$abc = -\frac{1}{2}(4)(-6) = 12$$